MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

A. K. Nandakumaran, Department of Mathematics, IISc, Bangalore

Problem set 6

1. Give an example to show that the conditions of integrability and σ-finiteness cannot be dropped in *Fubini's Theorem.*

2. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be two σ-finite spaces. Let $f : X \to \mathbb{R}$ be $\mathcal{F}$-measurable and $g : Y \to \mathbb{R}$ is $\mathcal{G}$-measurable. Show that $h : X \times Y \to \mathbb{R}$ defined by $h(x, y) = f(x)g(y)$ is $\mathcal{F} \times \mathcal{G}$ measurable.

3. Let $(X, \mathcal{F})$ be measurable space and $f : X \to \mathbb{R}$ is measurable function. Define

$$V^*(f) = \{(x, y) \in X \times \mathbb{R}, 0 \le y \le f(x)\}$$

and

$$V_*(f) = \{(x, y) \in X \times \mathbb{R}, 0 \le y < f(x)\}.$$

Prove

(a) If $f \ge 0$ and simple, then V^* and V_* are measurable in $X \times \mathbb{R}$.

(b) $V_*(f_n) \uparrow V_*(f)$ if $f_n \uparrow f$. If $f_n \downarrow f \ge 0$, then $V^*(f_n) \downarrow V^*(f)$.

(c) If $f \ge 0$, show that V^*, V_* are measurable in $X \times \mathbb{R}$.

(d) Let $G(f) = \{(x, y) : f(x) = y\}$ is the graph of f. Show that $G(f)$ is measurable in $X \times \mathbb{R}$.

(e) If μ is the Lebesque measure on $\mathbb{R}$, ν is the σ-finite measure on $(X, \mathcal{F})$. Assume $f \ge 0$. Show that $\mu \times \nu(V_*(f)) = \nu \times \mu(V^*(f)) = \int f d\mu$. (*This is a more precise formulation of the notion that the integral is the area under the curve.*)

4. Use *Fubini's Theorem* and other theorems on integration to integrate i) $\displaystyle\int_{\mathbb{R}} e^{x^2} dx = \sqrt{\pi}$,

ii) $\displaystyle\lim_{N \to \infty} \int_0^N \frac{\sin x}{x} = \frac{\pi}{2}$ (Hint: Use $\displaystyle\frac{1}{x} = \int_0^\infty e^{-xt} dt$)

5. i) Let $a_{i,j}$ be sequence of non-negative numbers, use *Fubini's Theorem* (Define appropriate measure spaces) to show that $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}$.

   ii) Let $a_{i,j}$ be any double sequence such that $\sum_{i,j} |a_{i,j}| < \infty$, then also prove that the above equality.

6. Let $X = Y = \mathbb{R}$, $\mu = \nu =$Lebesque measure. Then $\mu \times \nu$ is the two dimensional L-measure on $\mathbb{R} \times \mathbb{R}$, denote by $dxdy$ for $d(\mu \times \nu)$

   i) Let $E \subset \mathbb{R}$ measurable, show that $\sigma(E) = \{(x, y) : x - y \in E\}$ is measurable in $\mathbb{R}^2$. (*Hint: Prove it for open sets, G_δ sets, sets of measure zero and finally for general measurable sets.*)

   ii) Let f, g are measurable on $\mathbb{R}$. Show that $F(x, t) = f(x - t)g(y)$ is measurable in $\mathbb{R}^2$.

7. If g is measurable on [0,1], show that $f(x, y) = f(x) - g(y)$ is measurable on $[0, 1] \times [0, 1]$. Further, show that if f is integrable on $[0, 1] \times [0, 1]$, then g is integrable on [0,1].